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ON FURTHER DEVELOPMENT OF THE "METHOD OF LARGE λ" IN THE THEORY OF MIXED PROBLEMS

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The method of large λ [1], when the solution of the integral equations is represented as an asymptotic expansion in negative powers of some dimensionless parameter λ is used extensively, among the asymptotic methods of investigating the integral equations of the theory of mixed problems. As a rule only several terms of such an asymptotic expansion are constructed successfully.

Certain types of integral equations of the second kind, for which a method is proposed for the construction of all terms of the asymptotic expansion, are investigated below by the method of large λ. The coefficients and expansions of the required solution in negative powers of λ are represented as polynomials in the main argument and recursion formulas are obtained for the coefficients of these polynomials. Considered as examples are the axisymmetric mixed nonstationary problem of heat conduction for a homogeneous half-space and the axisymmetric problem of elasticity theory for the torsion of a truncated sphere by a press.

1. Solution of the integral equation. We examine the equation

$$\varphi(t) = \frac{1}{\pi\lambda} \int_{-1}^{1} \varphi(\tau) M\left(\frac{t-\tau}{\lambda}\right) d\tau + g(t) \quad (|t| \leqslant 1) \tag{1.1}$$

$$M(y)=\int_0^\infty [1-L(u)]\cos uy\,du=\sum_{k=0}^\infty b_k|y|^k \quad (|y|<B<\infty) \tag{1.2}$$

where $0<\lambda<\infty$ is a dimensionless parameter, $g(t)$ is a known function. The series (1.2) converge uniformly for $|y|<B<\infty$ (B is an arbitrarily large number).

We seek the solution of the integral equation (1.1) with kernel (1.2) in the form [2]

$$\varphi(t)=\sum_{n=0}^\infty \varphi_n(t)\lambda^{-n} \tag{1.3}$$

Substituting (1.3) and the series (1.2) into (1.1) and equating coefficients of identical powers of λ, we obtain the following recursion relations to determine $\varphi_n(t)$

$$\varphi_n(t)=\frac{1}{\pi}\sum_{i=0}^{n-1} b_i\int_{-1}^{1}\varphi_{n-i-1}(\tau)|t-\tau|^i\,d\tau \quad (n=1,2,\ldots), \qquad \varphi_0(t)=g(t) \tag{1.4}$$

We assume that $g(t)=1$. Then seeking $\varphi_n(t)$ in the form

$$\varphi_n(t)=\sum_{i=0}^{[n/2]}\eta_{n,i}t^{2i} \quad (n=1,2,\ldots) \tag{1.5}$$

(here and henceforth the square brackets in the limits of the summation denote the integer part of the number).

By substituting (1.5) into (1.4) and equating coefficients in identical powers of t^2 in the relationship obtained, we find recursion formulas to determine

$$\eta_{n,k}=\frac{2}{\pi}\sum_{l=1}^{k}(2l-1)!\,b_{2l-1}\eta_{n-l,k-l}\sum_{p=0}^{2l-1}\frac{(-1)^p(2k-2l+1+p)^{-1}}{(2l-1-p)!\,p!}+ \tag{1.6}$$
$$\frac{2}{\pi(2k)!}\sum_{l=2k}^{n-1}\frac{l!\,b_l}{(l-2k)!}\sum_{p=0}^{[(n-l-1)/2]}(2p+l-2k+1)^{-1}\eta_{n-l-1,p}$$

$$\eta_{n,0}=\frac{2}{\pi}\sum_{l=0}^{n-l}b_l\sum_{p=0}^{[(n-l-1)/2]}\eta_{n-l-1,p}(2p+l+1)^{-1} \tag{1.7}$$

$$\eta_{n\,[n/2]}=\frac{2}{\pi}\sum_{l=0}^{[n/2]}(2l-1)!\,b_{2l-1}\eta_{n-2l,[n/2]-l}\sum_{p=0}^{2l-1}\frac{(-1)^p(n-2l+1+p)^{-1}}{(2l-1-p)!\,p!} \tag{1.8}$$

where $k=1,2,\ldots,[n/2]$ in (1.6) if n is odd and $k=1,2,\ldots,[(n-1)/2]$ if n is even; n is even in (1.8).

A solution of (1.1) and (1.2) can be constructed in similar manner for large λ in the case when $g(t)=t^m$ (m is any natural number) as well as in the case when $g(t)$ is representable as a series in powers of the argument t.

Theorem 1. If $g(t)\in H_1^{1/2}(-1,1)$ and the inequality

$$\lambda>\lambda^\circ=\pi^{-1}[2b_0^*+d_1+\sqrt{(d_1+2b_0^*)^2+2d_2\pi}\,] \tag{1.9}$$

$$d_1=\sup_{|\varepsilon|\leqslant 1}\sqrt{\frac{2}{|\varepsilon|}}\int_{1/|\varepsilon|}^{\infty}|1-L(u)|\,du$$

$$d_2 = \sup_{|\varepsilon|\leqslant 1} \sqrt{2|\varepsilon|} \int_0^{1/|\varepsilon|} |1 - L(u)|\, du, \quad b_0^* = \int_0^\infty |1 - L(u)|\, du$$

is valid, then the solution of the integral equation (1. 1) with the kernel (1. 2) exists in the class $H_1^{1/2}(-1,1)$, is unique, and can be obtained by the method of large λ.

Here $H_1^{1/2}(-1,1)$ is the space of functions whose first derivative satisfies the Hölder condition with exponent $1/2$ on the segment $(-1,1)$.

If

$$M(y) = \sum_{n=0}^\infty b_n y^{2n} \quad (|y| < y_0) \tag{1.10}$$

$$g(t) = \sum_{n=0}^\infty a_n \lambda^{-2n} t^{2n} \quad (|t| < t_0) \tag{1.11}$$

in the integral equation (1. 1), then analogously to the preceding, the solution of (1. 1) with the kernel (1. 10) and right side (1. 11) can be represented for large values of λ as

$$\varphi(t) = \sum_{j=0}^\infty \lambda^{-2j} \sum_{i=0}^j \alpha_{ji} t^{2i} + \sum_{j=0}^\infty \lambda^{-(2j+1)} \sum_{i=0}^j \beta_{ji} t^{2i} \tag{1.12}$$

where α_{ji} and β_{ji} are determined from the following recursion relations:

$$\alpha_{ji} = \frac{2}{\pi(2i)!} \sum_{k=i}^{j-1} \frac{b_k (2k)!}{(2k-2i)!} \sum_{m=0}^{j-1-k} \frac{\beta_{j-1-k,\, m}}{2m+2k-2i+1} \quad (i = 0, 1, \ldots, j-1) \tag{1.13}$$

$$\beta_{ji} = \frac{2}{\pi(2i)!} \sum_{k=i}^{j} \frac{b_k (2k)!}{(2k-2i)!} \sum_{m=0}^{j-k} \frac{\alpha_{j-k,\, m}}{2m+2k-2i+1} \quad (i = 0, 1, \ldots, j)$$

$$\alpha_{jj} = a_j$$

2. Mixed problem of heat conduction. Let us examine the axisymmetric problem of heat conduction for a homogeneous half-space, when the temperature

$$T(r, 0, t) = T_0 e^{i\omega t} \quad (T_0 = \text{const}, \ \omega = \text{const}) \tag{2.1}$$

varying in a time t is given on the boundary in a circle of radius a , while there is no heat exchange on the rest of the surface.

The problem of determining the axisymmetric temperature field in cylindrical coordinates $T(r, z, t)$ in this case (we consider the process steady) is equivalent [2, 3] to the integral equation (1. 1), (1. 2) under the condition $g(t) = 1$,

$$L(u) = u\,[u^2 + (1+i)^2]^{-1/2}, \quad \lambda = \sqrt{2\varkappa}\,(a\sqrt{\omega})^{-1} \tag{2.2}$$

$$b_{2k+2} = -\sqrt{2}\, \frac{2^{k+1} i^{k+3/2}}{(2k+1)!!\,(2k+3)!!} \tag{2.3}$$

$$b_{2k+1} = -\frac{\pi i^{k+1}}{2^{k+1} k!\,(k+1)!}, \quad b_0 = -(1+i) \quad (k = 0, 1, \ldots)$$

($\varkappa$ is the coefficient of heat conduction).

Therefore, according to Sect. 1, the solution of the integral equation for the problem is representable as (1. 3), (1. 5) for the case of large λ where $\eta_{n,k}$ are determined from the recursion relations (1. 6) — (1. 8).

The magnitude of the total heat flux passing through the boundary of the half-space within the circle $r \leqslant a$ is determined for large λ by the formula

$$Q = 4aT_0\varkappa_0 e^{i\omega t} \sum_{n=0}^{N} \lambda^{-n} \sum_{i=0}^{[n/2]} \frac{\eta_{n,i}}{2i+1} \qquad (N \to \infty) \tag{2.4}$$

The solution of the problem posed here can also be obtained by the method of small λ elucidated in [2]. For small λ we have

$$\varphi(t) = aT_0 \frac{\pi}{4} F\left(\frac{1+t}{2}\right) F\left(\frac{1-t}{2}\right), \qquad F(x) = \Phi\left(-\frac{1}{2}, 1; -2ix\right) \tag{2.5}$$

$$Q = \pi a T_0 \varkappa_0 e^{i\omega t}\left(1 + \frac{1+i}{\lambda}\right) \tag{2.6}$$

($\Phi(-1/2, 1; -2ix)$ is the degenerate hypergeometric function). We introduce the quantities Q_1 and Q_2 by means of the formula

$$Q_1 + iQ_2 = Q\,(aT_0\varkappa_0)^{-1} e^{-i\omega t} \tag{2.7}$$

Results of numerical computations of the quantities Q_1 and Q_2 for various values of the parameter λ, computed by the method of large λ by means of (2.4) and (2.7) (columns 2 and 3) are presented in Table 1, where all the numbers presented are exact, and by the method of small λ by means of formulas (2.6) and (2.7) (columns 4 and 5).

Table 1

λ	Q_1	Q_2	Q_1	Q_2	λ	Q_1	Q_2	Q_1	Q_2
0.80	6.9244	3.9943	7.0686	3.9270	1.00	6.4876	3.0003	6.2832	3.1416
0.85	6.8172	3.6778	6.8376	3.6960	1.50	5.7070	1.8996	5.2360	2.0944
0.90	6.7054	3.4151	6.6323	3.4907	2.00	5.2828	1.5728	4.7124	1.5708
0.95	6.5946	3.1924	6.4485	3.3069					

As is seen from Table 1, the results of the method are in good agreement and permit making the deduction that the method of small λ should be used for the solution for $\lambda < 0.8$ and the method of large λ for $\lambda \geqslant 0.8$.

Note that the relative error is 0.001% for $\lambda = 0.8$ in the calculation of the quantities Q by the method of large λ if $N = 25$ in (2.4), i.e. if 25 terms of the series in λ are retained in (2.4), if $N = 10$ for $\lambda = 1.3$, and if $N = 4$ for $\lambda = 2.5$.

3. Torsion of a truncated sphere by a press. Let us consider the axisymmetric problem of elasticity theory concerning the torsion of a truncated sphere clamped rigidly to its plane boundary by a circular cylindrical press. We hence consider the spherical part of the sphere surface fixed.

This problem has been examined in [4, 5], where it was reduced to an integral equation of the second kind whose solution was not constructed. An approximate (closed) solution of the problem has been constructed in [2] on the basis of a special approximation of the kernel of the integral equation, and its asymptotic solution has been constructed in [6] in the case when the radius of the press is close to the radius of truncation.

An "exact" solution of the problem will be constructed below by the method elucidated in Sect. 1 for the case when the radius of the press is sufficiently smaller than the radius of the sphere truncated section.

The problem posed [2] can be reduced to the integral equation (1. 1) with the kernel (3. 1) and the right side of (3. 2) (c is a constant determined from the condition $\varphi(1)=0$)

$$M(y)=\int_0^\infty [1-\operatorname{th}\pi u \operatorname{th}\gamma u]\cos uy\,du \tag{3.1}$$

$$g(t)=\frac{\pi c_0}{2}\operatorname{ch}\frac{t}{2\lambda}-\sqrt{2}\operatorname{ch}^{-1}\frac{t}{2\lambda} \tag{3.2}$$

$$\frac{1}{\lambda}=2\operatorname{Arth}\frac{a}{b}=\alpha_0,\qquad \gamma=\arcsin\frac{b}{R}$$

Here $\gamma\in[0,\pi]$ is a parameter characterizing the degree of truncation of the sphere, a is the radius of the press, b is the truncation radius, and R is the radius of the sphere.

The kernel (3. 1) in the right side of (3. 2) can be expanded in the series (1. 10) and (1. 11), respectively, where

$$b_n=\frac{(-1)^n}{(2n)!}\int_0^\infty [1-\operatorname{th}\pi u\operatorname{th}\gamma u]\,u^{2n}du \tag{3.3}$$

$$a_n=c_0a_n{}^*-a_n{}^{**},\qquad a_0=\frac{\pi c_0}{2}-\sqrt{2}$$

$$a_n{}^*=\frac{\pi}{2(2n)!},\qquad a_n^{**}=\frac{4\sqrt{2}(-1)^n}{\pi^{2n+1}}\sum_{k=0}^\infty\frac{(-1)^k}{(2k+1)^{2n+1}}\qquad(n=1,2,\ldots)$$

Therefore, in the case of large values of λ the solution of the integral equation (1. 1) with the kernel (3. 1) and right side (3. 2) is representable as (1. 12) and (1. 13) under the condition (3. 3).

We introduce the notation

$$\varphi(t)=c_0\varphi^*(t)-\varphi^{**}(t)$$

which is related in a natural manner to the representation of the right side (3. 2) of the integral equation. Then from the condition $\varphi(1)=0$

$$c_0=\varphi^{**}(1)[\varphi^*(1)]^{-1}$$

For convenience in the numerical determination of the magnitude of the shear stresses under the press [2] by the method of large λ, let us represent them as

$$\tau_{\varphi z}(r,0)=-\frac{\sqrt{2}}{\pi}G\varepsilon\lambda(1+\operatorname{ch}\alpha)^{3/2}\operatorname{sh}\alpha\left\{\frac{\varphi'(1)}{\operatorname{sh}\alpha_0\sqrt{\operatorname{ch}\alpha_0-\operatorname{ch}\alpha}}-\right. \tag{3.4}$$

$$\frac{2\sqrt{\operatorname{ch}\alpha_0-\operatorname{ch}\alpha}}{\operatorname{sh}\alpha_0}\frac{d}{d\tau}\left[\frac{\varphi'(\tau\lambda)}{\operatorname{sh}\tau}\right]_{\tau=\alpha_0}+$$

$$\int_\alpha^{\alpha_0}\frac{d}{d\tau}\left[\frac{d}{d\tau}\left(\frac{\varphi'(\tau\lambda)}{\operatorname{sh}\tau}\right)\frac{1}{\operatorname{sh}\tau}\right]\sqrt{\operatorname{ch}\tau-\operatorname{ch}\alpha}\,d\tau$$

$$\varphi(\tau)=\sum_{i=0}^\infty\mu_i\tau^{2i},\qquad \mu_i=\sum_{j=i}^\infty\lambda^{-2j}(\alpha_{ji}+\lambda^{-1}\beta_{ji})$$

Here $\alpha=\operatorname{Arth}(r/b)$ (r is the distance between points of the half-space and the axis of symmetry), G is the shear modulus, and ε is the angle of press rotation.

The connection between the moment M acting on the press and the angle of stamp rotation ε for large λ is determined by the relationship

$$M = -2\sqrt{2}\,G\varepsilon a^2 \sum_{i=0}^{\infty} \lambda^{2i}\mu_i \int_0^{\alpha_0} \frac{t^{2i}\,dt}{\mathrm{ch}^3(t/2)}$$

Values of the dimensionless quantity

$$\tau^* = \tau_{\varphi z}(r, 0)\,(G\varepsilon)^{-1}$$

computed on an electronic digital computer by formulas (3.4), (1.12) and (1.13) to 0.01% accuracy, are presented in Table 2 for different γ and λ

Table 2

γ	$\frac{a}{b}$	r/b = 0.1	0.3	0.5	0.7	0.9
π	0.1	0.1280	0.4005	0.7352	1.248	2.629
	0.5	0.1308	0.4094	0.7521	1.279	2.698
	0.7	0.1356	0.4252	0.7840	1.341	2.864
2	0.1	0.1280	0.4005	0.7353	1.248	2.630
	0.5	0.1326	0.4152	0.7628	1.297	2.737
	0.7	0.1411	0.4425	0.8163	1.397	2.985
$\frac{\pi}{2}$	0.1	0.1280	0.4006	0.7354	1.249	2.630
	0.5	0.1350	0.4224	0.7758	1.318	2.779
	0.7	0.1447	0.4529	0.8329	1.426	3.191
1	0.1	0.1281	0.4009	0.7359	1.249	2.632
	0.5	0.1456	0.4545	0.8303	1.401	2.949
0.5	0.1	0.1287	0.4028	0.7394	1.255	2.644
	0.3	0.1509	0.4692	0.8507	1.418	2.921

If ρ denotes the shortest distance between points of the press for $r = a$ to the fixed spherical boundary of the sphere, then as computations show, the method of large λ yields a solution of the problem when the ratio ρ / a is sufficiently large.

The data in Table 2 are in sufficiently good agreement with the corresponding results presented in the tables of [2, 6] for this problem.

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INTEGRAL EQUATION OF THE INVERSE PROBLEM OF THE PLANE THEORY OF ELASTICITY

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Determination of the form of equal strength hole contours in a perforated plane under a specified load, i.e. the inverse problem, was formulated and solved with sufficient degree of generality by Cherepanov [1, 2] who reduced it to the Dirichlet problem for the exterior of a system of parallel slits in a plane, in the class of functions with power singularities at the ends of the slits, and a closed solution was obtained in a number of cases. In the present paper the initial problem for an arbitrary, finitely connected region is reduced to a Fredholm-type equation relative to the density of integral representation of the function which maps conformally a plane with circles excluded, onto a plane of the same connectivity with an unknown boundary. The equation obtained is solved by the method of least squares and this leads, in the case of any finitely connected region, to an unique computational scheme which can be programed into a computer. The coefficients of the corresponding algebraic system are determined and a one-parameter family of the contours sought is constructed for a plane, symmetrically periodic distribution of holes, as an example.

As we know [3], a canonical domain obtained from the ζ-plane by removing n circles, can be mapped onto any n-connected domain S_+ of the complex z-plane with a point at infinity. When $n > 2$, the mapping $\omega_0(\zeta)$ which has the form $\omega_0(\zeta) = C\zeta + \omega(\zeta)$, where $\omega(\zeta)$ is bounded at infinity, depends on $3n$ real parameters, six of which (e.g. one circumference, one fixed point on this circumference and a center of another circumference) can be specified arbitrarily, and C is a scale multiplier. Consequently, a system of contours of equal strength, if it exists, forms a $(3n - 6)$-parameter family. The limits of variation of the parameters can be found from geometrical considerations. The presence of symmetry may lead to reduction in the number of parameters.

We have the following relations [4] for determining the stress components at the boundary Γ of the region S_+:

$$\sigma_r + \sigma_\theta = 4\,\mathrm{Re}\,\Phi_0(\xi) \tag{1}$$

$$\sigma_\theta - \sigma_r + 2i\tau_{r\theta} = \frac{2(\xi - a_k)^2}{r_k^2\,\overline{\omega'(\xi)}}\left(\overline{\omega_0(\xi)}\,\Phi_0'(\xi) + \omega_0'(\xi)\,\Psi_0(\xi)\right) \tag{2}$$

Here σ_r, σ_θ and $\tau_{r\theta}$ denote the normal and shear stresses in a polar coordinate system with a pole at the center a_k of a circle of radius r_k and a boundary Γ_k, $k = 1, 2, \ldots, n$. If a homogeneous state of stress with the stress components σ_x, σ_y and τ_{xy} is given at infinity, then $\Phi_0(\zeta)$ and $\Psi_0(\zeta)$ have the form [4]